7 Singular support of Coherent sheaves

$$D(\mathrm{Bun}_G \Sigma) \cong QC(\mathrm{Flat}_{G^\vee}\Sigma)$$

is too naive

Perf

$\updownarrow$

Coh

$$D(\mathrm{Bun}_G \Sigma) \cong IC(\mathrm{Flat}_{G^\vee}\Sigma)$$

is also too naive

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1) Tangent complex

X space $\rightsquigarrow$ T_X tangent bundle

X alg var. $\xrightarrow[\text{if } X \text{ not smooth}]{}$ $\cancel{T_X}$ tangent sheaf $\in QC(X)^\heartsuit$

X alg var. $\rightsquigarrow$ $\mathbb{T}_X$ tangent complex

As always, begin with affine scheme

$S := \operatorname{Spec} A \in \mathrm{Sch}^{\mathrm{aff}}$ or $A \in \mathrm{ComAlg}^{\le 0}$

<u>Defn</u> In steps:

1. $\mathrm{Der}^i A = \{ \varphi : A \to A[i] \mid \varphi(fg) = \varphi(f)g + (-1)^{i|f|} f\varphi(g) \}$
2. $\mathrm{Der}^\bullet A$

   where $(d\varphi)(a) = d_A \varphi(a) + (-1)^{|\varphi|}\varphi(d_A a)$
3. $\mathbb{T}_S := \mathbb{T}_A := A \otimes_{\tilde{A}} \mathrm{Der}\,\tilde{A}$ where $\tilde{A}$ is quasi-free resolution

Ex] $A = k[x,y]/(xy)$

$\tilde{A} = k[x,y,\epsilon] \quad d\epsilon = xy\frac{\partial}{\partial\epsilon}$

is a $\underbrace{\ }_{\text{u.f.}}$ resolution of A

$$\tilde{A} = \quad \varepsilon k[x,y] \to k[x,y]$$
$$\epsilon \quad \to xy$$

rule $f\cdot g = (-1)^{|f||g|} g\cdot f$

$f = g = \epsilon \Rightarrow \varepsilon^2 = (-1)^{1\cdot 1}\varepsilon^2 \Rightarrow \varepsilon^2 = 0$

Facts |

- T_A is indep. of choice of $\tilde{A}$
- T_A is free A-module

eg. $A = k[x, \xi, \eta]$ $\quad |x| = 0$, $|\xi| = -2$, $|\eta| = -5$

T_A is of rk 1 in degrees $0, 2, 5$
$$\frac{\partial}{\partial x}\ \frac{\partial}{\partial \xi}\ \frac{\partial}{\partial \eta}$$

$$T_A = \Big(\overset{0}{\tilde{A} \oplus \tilde{A}} \to \overset{1}{\tilde{A}}\Big)$$
$$\frac{\partial}{\partial x}\quad \frac{\partial}{\partial y}\qquad \frac{\partial}{\partial \epsilon}$$

$\xi \to [d_A, \xi]$

$\frac{\partial}{\partial x} \to xy\frac{\partial}{\partial\epsilon}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}xy\frac{\partial}{\partial\epsilon} = -y\frac{\partial}{\partial\epsilon}$

$\frac{\partial}{\partial y} \to -x\frac{\partial}{\partial\epsilon}$

① $s = (x, y) \neq (0,0)$

$\dim H^0(\mathbb{T}_{s,s}) = 1$

② $0 = (0,0)$

$\dim H^0(\mathbb{T}_{s,0}) = 2$

$\dim H^1(\mathbb{T}_{s,0}) = 1$

$\Rightarrow \chi(\mathbb{T}_{s,s}) = 1$

$H_s = \mathbb{T}_s^\vee = \underline{Hom}_{QC(s)}(\mathbb{T}_s, \mathcal{O}_s)$

cotangent cpx

$H^i(\mathbb{T}_{s,s}[-1]) = H^{i-1}(\mathbb{T}_{s,s})$

Rmk] (shifted tangent cpx)

$\mathbb{T}_{s,s}[-1]$ has lie alg. str. in Vect

$\Omega_s S \to pt$ based loop space $\Rightarrow$ group object

$$\begin{tikzcd} \Omega_s S \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & S \end{tikzcd}$$

"Fiber product"

$\boxed{{}_s\Omega}\ \big|\ S$

In DAG, gp $\rightsquigarrow$ Lie alg.

$Lie(\Omega_s S) = \mathbb{T}_{s,s}[-1]$

Ex] $\mathbb{T}_{s,s}$: $\mathbb{C}^2 \to \overset{0}{\mathbb{C}}$

$s \neq 0$ $\quad \mathbb{C}^2 \to \mathbb{C} \simeq \overset{0}{\mathbb{C}}$

$s = 0$ $\quad \mathbb{C}^2 \to \mathbb{C}$

$\Omega_s S \cong k \otimes_A k$

(is not k.!) in derived setting

$\mathbb{T}_{s,s}[-1] = \underset{X, Y}{\mathbb{C}^2[-1]} \oplus \underset{Z}{\mathbb{C}[-2]}$

$[X, Y] = Z$

2) Quasi-smooth schemes and scheme of singularities

$\mathrm{Coh}\, X \xleftrightarrow{\text{different!}} \mathrm{Perf}\, X$

is from singular nature of X
not stacky nature

Goal: Find a reasonable class of singular schemes

Prop1: A derived scheme Z is smooth classical
$\Leftrightarrow \mathbb{T}_Z$ is a vector bundle
$\Leftrightarrow H^i(\mathbb{T}_{Z,z}) = 0 \quad \forall i > 0,\ z \in Z$

Defn1: A derived scheme is quasi-smooth
if $\mathbb{T}_Z$ is perfect of amplitude $[0,1]$
($\Leftrightarrow H^i(\mathbb{T}_{Z,z}) = 0 \quad \forall i > 1,\ z \in Z$)

$$\mathbb{T}_Z|_U = \left(\mathcal{O}_Z^n|_U \to \mathcal{O}_Z^m|_U [-1] \right)$$

Rmk1: $\mathcal{X}$ a moduli space
$\Rightarrow$ want intersection theory of $\mathcal{X}$
For that, we need $[\mathcal{X}]^{vir}$
In all the cases appearing in enumerative geometry,
$[\mathcal{X}]^{vir}$ arises from quasi-smoothness of $\mathcal{X}^{der}$ derived version of $\mathcal{X}$ "perfect obstruction theory"

(it is believed)

Prop) A derived scheme Z is quasi-smooth iff Z can be written (Zariski-locally) as

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "r"] & \mathbb{A}^n \arrow[d, "F"] \\ pt \arrow[r, "\{0\}"] & \mathbb{A}^m \end{tikzcd}$$

pf. $\Leftarrow$)

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & U \arrow[d] \\ pt \arrow[r] & V \end{tikzcd}$$

U, V $\overset{\text{smooth}}{\wedge}$ classical schemes

$$\mathbb{T}_Z = \ker(df : \mathbb{T}_U|_Z \to \mathbb{T}_V|_Z)$$

$$\Rightarrow \mathbb{T}_Z = \ker(\mathcal{O}_Z^n \to \mathcal{O}_Z^m) \quad \text{Zariski-locally}$$

$$= \mathcal{O}_Z^n \to \mathcal{O}_Z^m[-1]$$

$\boxtimes$

- $m=1$ $\rightsquigarrow$ hypersurface

In particular, all hypersurfaces are quasi-smooth

- A qs derived scheme $\Leftrightarrow$ locally complete intersection in a derived world
- A qs classical scheme $\Leftrightarrow$ l.c.i. from regular sequence

$$\mathbb{T}_Z = (\mathbb{T}_U|_Z \xrightarrow{df} \mathbb{T}_V|_Z[-1])$$

$$\mathbb{L}_Z = (\mathbb{L}_V|_Z[1] \xrightarrow{df^*} \mathbb{L}_U|_Z)$$

If Z is smooth, df is surjective (df^* is inj) we only have $H^0(\mathbb{T}_Z)$, $H^0(\mathbb{L}_Z)$

If Z is not, we have $H^1(\mathbb{T}_Z)$, $H^{-1}(\mathbb{L}_Z)$

Z quasi smooth $\boxed{\text{classical}}$

$\rightsquigarrow$ $\mathrm{Sing}\, Z$ scheme of singularities

s.t. $\mathrm{Sing}\, Z$ measures how far Z is from being smooth

<u>Defn</u>

$$\left.\begin{aligned} \mathrm{Sing}\, Z &= \mathrm{Spec}_{Z^{cl}} \mathrm{Sym}_{H^0(\mathcal{O}_Z)} H^1(T_Z) \\ &= (T^*[-1]Z)^{cl} \\ \mathcal{O}_{T^*[n]Z} &= \mathrm{Sym}_Z\, T_Z[-n] \end{aligned}\right\} \text{for } Z \text{ quasi-smooth}$$

$$\begin{tikzcd} \mathrm{Sing}\, Z \arrow[d] \\ Z^{cl} \end{tikzcd}$$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^n \arrow[d] \\ pt \arrow[r, "\{0\}"] & \mathbb{A}^m \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d] \\ pt \arrow[r, "\{0\}"] & V \end{tikzcd} \qquad T_{V,pt} = V \qquad \mathbb{L}_V|_Z \cong \mathcal{O}_Z \otimes V^*$$

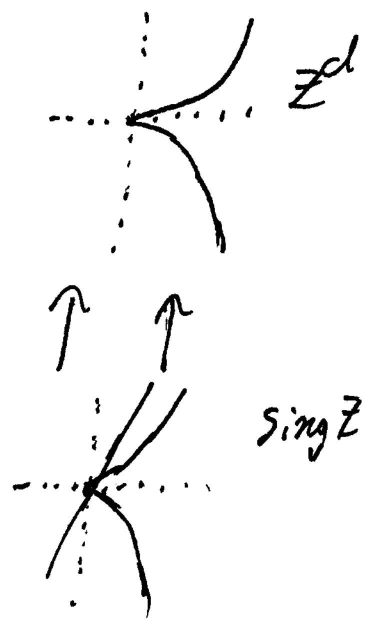
$$\Rightarrow \mathrm{Sing}\, Z \subset Z^{cl} \times V^*$$

$$\begin{tikzcd} \mathrm{Sing}\, Z \arrow[d] \\ Z^{cl} \end{tikzcd}$$

$f = y^2 - x^3$

<u>Ex</u> $U \xrightarrow{f} \mathbb{A}^1$

$$\begin{tikzcd} Z = Z(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d, "f"] \\ \{0\} \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$



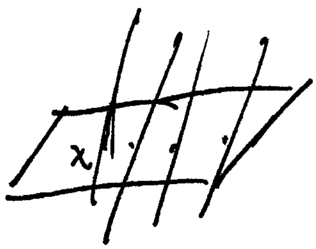
3) Singular support of coherent sheaves

$\mathrm{Coh}(X)$ vs. $\mathrm{Perf}(X)$

A classical associative alg $\quad A\text{-mod}^{\heartsuit}$

$U(\mathfrak{g})\text{-mod}$

$A\text{-mod}$



$\mathrm{Spec}(Z\mathfrak{g})$

$\mathrm{Spec}(ZA)$

Let $\mathcal{C}$ be a DG category.

Defn) The center of $\mathcal{C}$ is

$$HC(\mathcal{C}) = \mathrm{End}(\mathbb{1}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C})$$

$\uparrow$

"Hochschild cochains"

$\{\varphi_c \in \mathrm{Hom}(c,c)\}_{c \in \mathcal{C}}$ s.t. for $c \xrightarrow{f} c'$; $f \circ \varphi_c = \varphi_{c'} \circ f$

Ex) $\mathcal{C} = A\text{-mod}$

$\{\varphi_M\}_{M \in A\text{-mod}} \longleftrightarrow \varphi_A : A \to A \in \mathrm{End}_A A$ ~~[illegible]~~

central as we consider

$$f_a : A \to A$$
$$\overset{\shortparallel}{a\cdot}$$
$$\mathrm{End}_A(A)$$

$$HH^\bullet(\mathcal{C}) = \bigoplus_{n\in\mathbb{Z}} H^n(HC(\mathcal{C}))$$

Hochschild coh of $\mathcal{C}$

$$HH^0(A\text{-mod}) \to Z(\mathrm{End}_A(A))$$

$$= Z(A^{op}) = Z(A)$$

$\mathcal{C}$ DG category $\rightsquigarrow T = Ho(\mathcal{C})$

$\boxed{R \circlearrowright \mathcal{C} \quad (\text{e.g. } \overset{\shortparallel}{HC(\mathcal{C})})}$

$R \circlearrowright T$

graded comm. algebra

$r \in R^{2n}$ $\quad r\cdot t \to t[2n]$ $\quad \forall t \in T$

$\forall f : t \to t'$,

$$\begin{tikzcd} t \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & t' \arrow[d] \\ t[2n] \arrow[r, "f"] & t[2n] \end{tikzcd}$$

$R \to HH^\bullet(\mathcal{C}) \circlearrowright T$

$\uparrow$

we want to find this!

<u>Thm</u> (Hochschild Konstant Rosenberg)

let X smooth affine scheme $/k$, char $k = 0$

$$HH^\bullet(QC(X)) = H^0(X, \Lambda^\bullet_{\mathcal{O}} T_X)$$

note $QC(X) = \mathcal{O}_X\text{-mod}$ $\qquad$ polyvector fields

$= A\text{-mod}$

$HC(A) = Ext(A,A) = A\otimes A^{op}$

X quasi-smooth affine

$$HC(X) \simeq \Gamma(X, \mathcal{U}_{\mathcal{O}_X}(\mathbb{T}_X[-1]))$$

as associative algebra

$\| \; HC(QC(X))$

$\mathbb{T}_X[-1]$: Lie algebra in $QC(X)$

$\mathcal{U}$ universal enveloping algebra

If X is smooth, $\mathbb{T}_X = T_X$

$$\mathcal{U}_{\mathcal{O}_X}(\mathbb{T}_X[-1]) = \mathrm{Sym}_{\mathcal{O}_X} T_X[-1] =: \Lambda^\bullet_{\mathcal{O}_X} T_X$$

trivial lie alg str

symmetric + deg 1 shifting

$\Gamma(X, \mathcal{O}_X) \to HC(X)$ module over

$\Gamma(X, \mathbb{T}_X[-1]) \to HC(X)$

$H^0(X, \mathcal{O}_X) \to HH^0(X)$ module

quasi-smooth $H^1(X, \mathbb{T}_X) \to HH^2(X)$

$:= \mathrm{Sym}_{H^0(X,\mathcal{O}_X)}[H^1(X,\mathbb{T}_X)] \to HH^\bullet(X) \to \mathrm{End}(\mathcal{F})$

$\forall \mathcal{F} \in Coh(X)$

Defn: $\mathcal{F} \in Coh(X)$

- $\mathrm{Sing\,Supp}\,\mathcal{F} = \mathrm{supp}_{\mathrm{Sing}\,X} \mathrm{End}\,\mathcal{F} \subset \mathrm{Sing}\,X$
- for $Y \subset \mathrm{Sing}\,X$, $Coh_Y(X) \subset Coh(X)$ is full subcategory consisting of sheaves $\mathcal{F}$ s.t. $\mathrm{Sing\,supp}\,\mathcal{F} \subset Y$.

$$\begin{tikzcd}
 & QC(\mathrm{Flat}_{G^\vee}) \arrow[dl, bend right, "\neq"'] \arrow[d, phantom, "\subset" rotate=-90] & 0 \arrow[d, phantom, "\subset" rotate=-90] \\
D(\mathrm{Bun}_G) \arrow[r, "{[A\text{-}G]}", "\sim"'] \arrow[dr, bend right, "\neq"] & IC_N(\mathrm{Flat}_{G^\vee}) \arrow[d, phantom, "\subset" rotate=-90] & N/G \arrow[d, phantom, "\subset" rotate=-90] \\
 & IC(\mathrm{Flat}_{G^\vee}) & \mathfrak{g}^*/G
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & U \arrow[d] \\
pt. \arrow[r] & V
\end{tikzcd} \rightsquigarrow \mathrm{Sing}\, X \subset X \times V^*$$

$\mathrm{Loc}_G C$ moduli of local systems

$\overset{\shortparallel}{\mathrm{Hom}}(\pi_1(C), G)/G$

$xyx^{-1}y^{-1}$

$$\begin{tikzcd}
\mathrm{Hom}(\pi_1(C), G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G^{2g} = U \arrow[d, "{[\,,\,]}"] \\
\{1\} \arrow[r] & G = V
\end{tikzcd}$$

$\mathrm{Sing}\ \mathrm{Loc}_G = \mathrm{Loc}_G \times \mathfrak{g}^*/G$

$N_G = \mathrm{Loc}_G \times N$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{A}^n \arrow[d] \\ \{pt\} \arrow[r] & \mathbb{A}^m \end{tikzcd} \qquad n=0$$

$W = \operatorname{Spec} k[\eta_1, \cdots, \eta_m] \qquad |\eta_i| = -1$

<u>Thm</u> (Koszul duality)

① $\operatorname{Ext}_{k[\eta]}(k,k) = k[\xi_1, \cdots, \xi_m] \qquad |\xi_i| = 2$

② $K : k[\eta]\text{-mod} \to k[\xi]\text{-mod}$

$\qquad M \to \underline{\operatorname{Hom}}_{k[\eta]}(k, M)$

induces a fully faithful functor on $k[\eta]^{f.g.}$-mod

③ $\operatorname{Coh}(W) \simeq k[\xi]\text{-mod}$

$$\begin{tikzcd} \operatorname{Perf}(W) \arrow[r, hook] \arrow[d] & QC(W) \arrow[d] \\ \operatorname{Coh}(W) \arrow[r, hook] & IC(W) \end{tikzcd} = \begin{tikzcd} k[\xi]\text{-mod}_0^{f.g.} \arrow[r, hook] \arrow[d] & k[\xi]\text{-mod}_0 \arrow[d] \\ k[\xi]\text{-mod}^{f.g.} \arrow[r, hook] & k[\xi]\text{-mod} \end{tikzcd}$$